Intro Video: Section 4.4
Limits of indeterminate type and L'Hôpital's Rule

Math F251X: Calculus I

Limits of indeterminate type

Compare

$$\lim_{x\to 3^-} \frac{x^2-5}{x+2} = \frac{4}{5}$$

Try direct substitution?

$$\frac{9-5}{3+2} = \frac{4}{5}$$

$$\lim_{x\to 3^-} \frac{x^2-5}{x-3} = -\infty$$

$$\frac{9-5}{3-3} = \frac{9}{0}$$

$x \to 3$

$x-3<0$

$$\lim_{x\to 3^-} \frac{x^2-x-6}{x-3}$$

$$\frac{9-3-6}{3-3} = \frac{0}{0}$$

Compare

$$\lim_{x\to 3^-} \frac{\ln(4-x)}{x-3}$$

direct sub?

$$\frac{\ln(4-3)}{3-3} = \frac{\ln(1)}{0}$$

$$= \frac{0}{0}$$

What does "indeterminate form" mean?

→ When we "plug in" for x, we get something that looks like...

$$\frac{0}{0} \qquad \frac{\infty}{\infty} \qquad \infty - \infty \qquad 0 \cdot \infty \qquad 1^{\infty} \qquad 0^{0} \qquad \infty^{0}$$

Not indeterminate forms:

$$\infty + \infty \qquad 1^{0} \qquad \frac{1}{\infty}, \frac{\#}{\infty} \qquad \underbrace{\frac{1}{0}, \frac{\#}{0}}_{\text{know how to assess behavior (but may need to look at both sides)}} \qquad 0^{\infty} \qquad \infty^{\infty}$$

How to evaluate limits of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$:

L'Hôpital's Rule. (a.k.a L'Hospital's Rule)

L'Hôpital's Rule: If a limit has the form (indeterminate type) $0/0$ or $\frac{\infty}{\infty}$, then

$$\lim_{x \to *} \frac{f(x)}{g(x)} \overset{\boxed{L'H}}{=} \lim_{x \to *} \frac{f'(x)}{g'(x)}.$$

as long as $\lim_{x \to *} \frac{f'(x)}{g'(x)}$ exists or equals $\pm\infty$.

$*$ can be a, a^-, a^+, ∞

Example:

$$\lim_{x \to 3^-} \frac{x^2 - x - 6}{x - 3}$$

$$\boxed{L'H} = \lim_{x \to 3^-} \frac{2x - 1}{1}$$

$$= 2(3) + 1$$

$$= 5.$$

$$\text{type} = \frac{3-3-6}{3-3} = \frac{0}{0}$$

Chapter 2 techniques:

$$\lim_{x \to 3^-} \frac{x^2 - x - 6}{x - 3}$$

$$= \lim_{x \to 3^-} \frac{(x-3)(x+2)}{x-3}$$

$$= \lim_{x \to 3^-} (x+2)$$

$$= 3 + 2 = 5$$

Be careful!

$$\lim_{x \to 3^-} \frac{x^2-5}{x+2} = \frac{9-5}{3+2} = \frac{4}{5}$$

$$\overset{L'H}{=} \lim_{x \to 3^-} \frac{2x}{1}$$

$$= 2(3)$$

$$= 6$$

If you use L'H on a function that is NOT indeterminate type, You will get out garbage!

Always check your type before using L'Hôpital's Rule!

Example

$$\lim_{t \to 0} \frac{e^{2t} - 1}{\sin(t)}$$

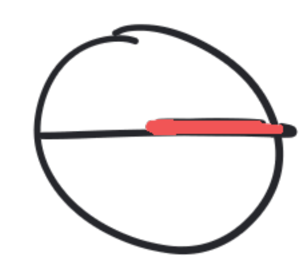
type: $\frac{e^0 - 1}{\sin(0)} = \frac{0}{0}$

$\boxed{L'H}$

$$= \lim_{t \to 0} \frac{e^{2t}(2)}{\cos(t)}$$

$$= \frac{e^{2(0)} \cdot 2}{\cos(0)} = \frac{2}{1}$$

$$= 2$$



Example:

$$\lim_{x \to \infty} \frac{(\ln(x))^2}{x} \quad \text{type } \frac{\infty}{\infty}$$

$$\overset{\boxed{L'H}}{=} \lim_{x \to \infty} \frac{2\ln(x) \cdot \frac{1}{x}}{1}$$

$$= \lim_{x \to \infty} \frac{2\ln(x)}{x} \quad \text{type } \frac{\infty}{\infty}$$

$$\overset{\boxed{L'H}}{=} \lim_{x \to \infty} \frac{2 \cdot 1/x}{1}$$

$$= \lim_{x \to \infty} \frac{2}{x}$$

$$= 0$$

Example:

$$\lim_{x \to \infty} x \sin\left(\frac{\pi}{x}\right)$$

type

$\infty \cdot 0$

$$= \lim_{x \to \infty} \frac{\sin(\pi/x)}{1/x}$$

$x = \dfrac{1}{\frac{1}{x}}$

$\dfrac{0}{0}$

$\boxed{L'H}$

$$= \lim_{x \to \infty} \frac{\cos\left(\frac{\pi}{x}\right)\left(-\pi x^{-2}\right)}{-x^{-2}}$$

$$= \lim_{x \to \infty} \frac{-\frac{\pi}{x^2} \cos\left(\frac{\pi}{x}\right)}{-\frac{1}{x^2}}$$

$$= \lim_{x \to \infty} -\pi \cos\left(\frac{\pi}{x}\right) = -\pi \cos(0) = -\pi.$$

Example:

$$\lim_{x \to 1} \frac{x}{x-1} - \frac{1}{\ln(x)}$$

type

$$\frac{1}{0} - \frac{1}{0} \leftarrow \infty - \infty$$

$$= \lim_{x \to 1} \frac{x\ln(x) - 1(x-1)}{(x-1)\ln(x)}$$

type $\frac{1(0)-1(0)}{0(0)} = \frac{0}{0}$

$$\overset{\boxed{L'H}}{=} \lim_{x \to 1} \frac{\left[x \cdot \frac{1}{x} + \ln(x)\right] - 1}{(x-1)\cdot\frac{1}{x} + \ln(x)(1)}$$

$$= \lim_{x \to 1} \frac{1 + \ln(x) - 1}{1 - 1/x + \ln(x)}$$

$$\frac{1 + \ln(1) - 1}{1 - 1 + \ln(1)} = \frac{0}{0}$$

$$\overset{\boxed{2' H}}{=} \lim_{x \to 1} \frac{1/x}{-\left(\frac{-1}{x^2}\right) + \frac{1}{x}} = \frac{1/1}{\left(\frac{1}{1}\right)^2 + \frac{1}{1}} = \frac{1}{2}$$

Indeterminate form	technique	**NOT indeterminate forms**	limit
$\frac{0}{0}$	Algebra; L'H if necessary	$\infty + \infty$	∞
$\frac{\infty}{\infty}$	Algebra; L'H if necessary	1^0	1
$\infty - \infty$	algebra to rewrite as $\frac{0}{0}$ or $\frac{\infty}{\infty}$	$\frac{1}{\infty}$	0
$0 \cdot \infty$	algebra to rewrite as $\frac{0}{0}$	$\infty \cdot \infty$ and ∞^∞	∞
1^∞	Use logs to transform	$\frac{1}{0}$ (May not exist)	$\pm\infty$
0^0	Use logs to transform	0^∞	0
∞^0	Use logs to transform	∞^∞	∞

$\infty \cdot (-\infty) = -\infty$

<u>Example:</u> $\lim_{x\to 0^+} x^{\sqrt{x}}$

Let $y = x^{\sqrt{x}}$.

$$\lim_{x\to 0^+} \ln(y) = \lim_{x\to 0^+} \ln\left(x^{\sqrt{x}}\right) = \lim_{x\to 0^+} \sqrt{x}\cdot\ln(x)$$

$0\cdot(-\infty)$

$$= \lim_{x\to 0^+} \frac{\ln(x)}{x^{-1/2}}$$

type $\frac{-\infty}{\infty}$; $x^{-1/2} = \frac{1}{\sqrt{x}}$.

$$\overset{\boxed{L'H}}{=} \lim_{x\to 0^+} \frac{1/x}{-1/2\,x^{-3/2}} = \lim_{x\to 0^+} \frac{-x^{3/2}}{2x} = \lim_{x\to 0^+} \frac{-x^{1/2}}{2} = 0$$

$$\lim_{x\to 0} \ln(y) = 0 \Rightarrow \lim_{x\to 0} x^{\sqrt{x}} = e^0 = 1.$$